Exceptionally simple super-PDE

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A Quantum and Super Afternoon
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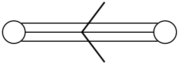
Joint work with B. Kruglikov & D. The (Adv. Math. **376** (2021), 98 pp.)
and joint work with D. The (preprint arXiv:2207.04531v1 (2022), 37 pp.)

Plan of the talk:

- Prelude: a G_2 story
- The Lie superalgebra $G(3)$: parabolic subalgebras & Spencer cohomology
- Realizations of $G(3)$ as supersymmetry of geometric structures
- Latest developments: the mixed contact and the odd-contact $F(4)$ results

Some geometric realizations of G_2

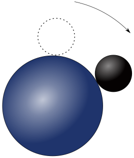


This is an abstract description via Dynkin diagrams. What about realizations as symmetries?

– $GL_7(\mathbb{C})$ acts with open orbit on 3-forms on $\mathbb{C}^7$ and $G_2 = \mathrm{Stab}_{GL_7(\mathbb{C})}(\phi)$ for generic $\phi \in \bigwedge^3(\mathbb{C}^7)^*$ (Engel, 1900);

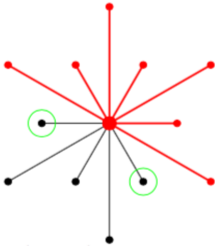
– Compact form $G_2 = \mathrm{Aut}(\mathbb{O})$ (Cartan, 1914);

– Configuration space M of a 2-sphere rolling on another w/o twisting or slipping is 5-dimensional, with the constraints given by a rank 2 distribution $\mathcal{D} \subset \mathcal{T}M$ of filtered growth $(2, 3, 5)$. If the ratio of the radii of spheres is 3, then split $G_2 = \mathrm{Aut}(M, \mathcal{D})$ (Bryant, Zelenko, Bor–Montgomery, Baez–Huerta).



$(2,3,5)$-geometry from the G_2 root diagram

G_2/P_1



Fundamental invariant of $(2,3,5)$-distributions: binary quartic field (Cartan 1910).
Modern perspective: the quartic arises from $H^{4,2}(\mathfrak{m},\mathfrak{g}) \cong S^4(\mathbb{C}^2)$, where $\mathfrak{g} = G_2$ has $|3|$-grading $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ with negative part

$$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5 \rangle$$

$$[e_1, e_2] = e_3 \ , \qquad [e_1, e_3] = e_4 \ , \qquad [e_2, e_3] = e_5 \ ,$$

and 0-degree component $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m}) \cong \mathfrak{gl}(2)$.

Some geometric realizations of G_2

– Engel (1893): G_2 as symmetry of contact distribution $\mathcal{C}$ on 5-dim. mnfd with field of twisted cubics $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$;

– Cartan (1893, 1910): G_2 as symmetry of

Dim	Geometric structure	Model
5	ODE with flat $(2,3,5)$-distribution	$du - u'dx$, $du' - u''dx$, $dz - (u'')^2 dx$, *Hilbert–Cartan equation* $z' = (u'')^2$
6	Pair of PDE (with flat contact distribution)	$u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$

Today: realizations of the Lie superalgebra $G(3) = \big(G_2 \oplus \mathfrak{sp}(2)\big) \oplus \big(\mathbb{C}^7 \otimes \mathbb{C}^2\big)$.

Main Motivations and Goals

General motivations.

- Give geometric realizations of Lie superalgebras $G(3)$, $F(4)$, $\mathfrak{osp}(4|2;\alpha)$ as symmetry superalgebras of simple objects;
- We are interested in geometries that have high symmetry, a lot of solns to BGG eqns, Killing spinor eqns, etc. For example, we plan to understand relationship symmetries of superdistributions $\leftrightsquigarrow$ supergravity bkgds;
- Here is another suggestion: does any given classical geometry admit a non-trivial supersymmetric extension?

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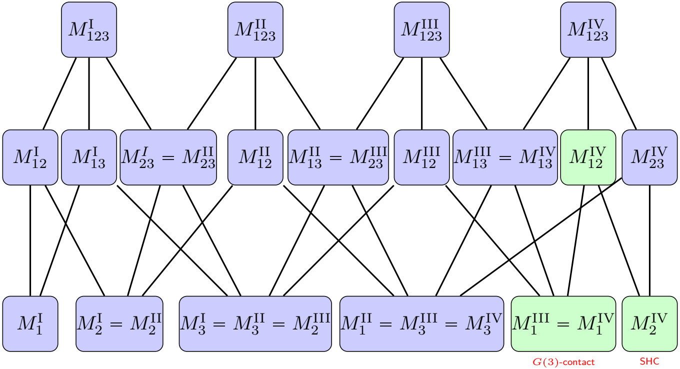
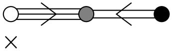
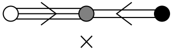
Goals achieved so far.

- Various geometric realizations of $G(3)$;
- Understanding of the deformations of these flat structures;
- In particular, we exhibited superextensions of the flat and some non-flat $(2, 3, 5)$-geometries, and gave bounds on supersymmetry dimension.

Map of $G(3)$-supergeometries



We considered two $\mathbb{Z}$-gradings of $\mathfrak{g} = G(3)$ with (graded) growths:

	marked Dynkin diagram	$\dim(\mathfrak{g}_{-1})$	$\dim(\mathfrak{g}_{-2})$	$\dim(\mathfrak{g}_{-3})$
$G(3)$-contact		$4\|4$	$1\|0$	
SHC		$2\|4$	$1\|2$	$2\|0$

Geometric structures associated to M_1^{IV} and M_2^{IV}

G(3)-contact super-PDE:

$$u_{xx}=\frac{1}{3}(u_{yy})^3+2u_{yy}u_{y\nu}u_{y\tau}, \quad u_{xy}=\frac{1}{2}(u_{yy})^2+u_{y\nu}u_{y\tau},$$
$$u_{x\nu}=u_{yy}u_{y\nu}, \quad u_{x\tau}=u_{yy}u_{y\tau}, \quad u_{\nu\tau}=-u_{yy}.$$

where $u=u(x,y|\nu,\tau):\mathbb{C}^{2|2}\to\mathbb{C}^{1|0}$.

Super Hilbert-Cartan equation (SHC):

$$z_x=\frac{(u_{xx})^2}{2}+u_{x\nu}u_{x\tau}, \quad z_\nu=u_{xx}u_{x\nu}, \quad z_\tau=u_{xx}u_{x\tau}, \quad u_{\nu\tau}=-u_{xx},$$

where $(u,z)=(u(x|\nu,\tau),z(x|\nu,\tau)):\mathbb{C}^{1|2}\to\mathbb{C}^{2|0}$.

Thm[Kruglikov, S., The] These super-PDE have symmetry superalgebras $G(3)$. Unlike the Hilbert-Cartan eqn, whose general solution depends on one arbitrary function of one variable, solutions of SHC depend only on five constants.

Tanaka–Weisfeiler prolongation and Spencer cohomology

Given negatively graded Lie superalgebra $\mathfrak{m} = \mathfrak{m}_{-\mu} \oplus \cdots \oplus \mathfrak{m}_{-1}$ and $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{m})$, we let the Tanaka–Weisfeiler prolongation $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ be graded Lie superalgebra s.t.:

(i) $\mathrm{pr}_{\leqslant 0}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$;

(ii) if $[X, \mathfrak{g}_{-1}] = 0$ for $X \in \mathrm{pr}_+(\mathfrak{m}, \mathfrak{g}_0)$ then $X = 0$;

(iii) $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ is maximal with these properties.

If $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m})$, we simply write $\mathrm{pr}(\mathfrak{m})$.

Rem I. Although $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ can be obtained via an iterative process, one can test a candidate $\mathfrak{g}$ that extends $\mathfrak{m} \oplus \mathfrak{g}_0$ via the criteria:

- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m})$ if and only if $H^1_{\geqslant 0}(\mathfrak{m}, \mathfrak{g}) = 0$;
- $\mathfrak{g} = \mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$ if and only if $H^1_+(\mathfrak{m}, \mathfrak{g}) = 0$.

Rem II. Kostant's version of BBW Thm efficiently computes these groups in the classical setting but in super-setting his "harmonic cohomology" is usually bigger.

Spencer cohomology of SHC grading

Thm[Kruglikov, S., The] Let $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \cdots \oplus \mathfrak{g}_3$ be the SHC grading of $\mathfrak{g} = G(3)$. Then $H^{d,1}(\mathfrak{m}, \mathfrak{g}) = 0$ for all $d \geqslant 0$, so that $\mathfrak{g} \cong \mathrm{pr}(\mathfrak{m})$. Moreover $H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{1}} = 0$ for all $d > 0$ while

$$H^{d,2}(\mathfrak{m}, \mathfrak{g})_{\bar{0}} \cong \begin{cases} 0 & \text{for all } d > 0, d \neq 2, \\ S^2\mathbb{C}^2 \boxtimes \Lambda^2\mathbb{C}^2 & \text{if } d = 2, \end{cases}$$

Rem I. As a $(\mathfrak{g}_0)_{\bar{0}}$-module, the space $C^{4,2}(\mathfrak{m}, \mathfrak{g})$ has a unique submodule $S^4\mathbb{C}^2 \boxtimes \mathbb{C}$, which is the space of Cartan's classical binary quartic invariants. Its elements are not closed in the complex $C^\bullet(\mathfrak{m}, \mathfrak{g})$.

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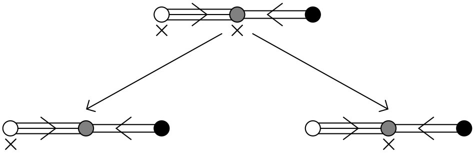
Rem II. This suggests the Cartan quartic of underlying generic rank 2 distribution on 5-dim. mnfd should admit a square root, hence it must be of Petrov type D (pair of double roots), N (quadruple root) or O (identically zero).

$G(3)$-double fibration

We investigated the $G(3)$-twistor correspondence



$G(3)$-contact SHC

Strategy: flag supermnfd $G(3)/P_1$ is contact supermnfd $(M, \mathcal{C})$ with the additional reduction of structure group $COSp(3|2) \subset CSpO(4|4)$, which we realize as $(1|2)$-twisted cubic $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$. Osculate $\mathcal{V}$ to get PDE $\mathcal{E} \cong G(3)/P_{12}$. Cartan superdistrib. of $\mathcal{E}$ has "Cauchy characteristic", we quotient by it to get SHC eqn $\overline{\mathcal{E}} \cong G(3)/P_2$.

$G(3)$-contact case

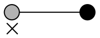
Idea: contact supermnfd + additional geometric structure.

k	$(\mathfrak{g}_k)_{\bar{0}}$	$(\mathfrak{g}_k)_{\bar{1}}$	dim
0	$\mathbb{C} \oplus \mathfrak{sl}(2) \oplus \mathfrak{sp}(2)$	$S^2\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$7\|6$
-1	$S^3\mathbb{C}^2 \boxtimes \mathbb{C}$	$\mathbb{C}^2 \boxtimes \mathbb{C}^2$	$4\|4$
-2	$\mathbb{C} \boxtimes \mathbb{C}$		$1\|0$

Prop. $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{osp}(3|2) \subset \mathfrak{der}_{gr}(\mathfrak{m}) = \mathbb{C} \oplus \mathfrak{spo}(4|4)$ is a maximal subalgebra.

The basis of $V := \mathfrak{g}_{-1}$ given by $\{x^3, x^2y, xy^2, y^3 | xe, xf, ye, yf\}$ allows to make explicit the invariant $CSpO$-structure on V. The topological point $[x^3] \in \mathbb{P}(V_{\bar{0}})$ has isotropy $\mathfrak{q} \subset \mathfrak{f} := \mathfrak{osp}(3|2)$ that is a parabolic subalgebra:

$$\mathfrak{f} = \mathfrak{f}_{-1} \oplus \overbrace{\mathfrak{f}_0 \oplus \mathfrak{f}_1}^{\mathfrak{q}}$$



k	$\mathfrak{f}_k$
1	$(\mathbb{C}^{1\|2})^*$
0	$\mathbb{C} \oplus \mathfrak{osp}(1\|2)$
-1	$\mathbb{C}^{1\|2}$

The $(1|2)$-twisted cubic $\mathcal{V}$

Def. The $COSp(3|2)$-orbit $\mathcal{V} \subset \mathbb{P}(V)$ through $[x^3]$ is called $(1|2)$-twisted cubic.

We describe $\mathcal{V}$ locally by exponentiating the action of $\mathfrak{f}_{-1} = \operatorname{span}\{Y|A, B\} \cong \mathbb{C}^{1|2}$ through $[x^3]$:

$$\left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right) \overset{\exp(\lambda Y)}{\longmapsto} \left(\begin{array}{c} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right) \overset{\exp(\theta A)}{\longmapsto} \left(\begin{array}{c} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} \\ -\frac{\lambda^2}{2} \\ \hline \theta \\ 0 \\ 0 \\ -\theta\lambda \end{array}\right) \overset{\exp(\phi B)}{\longmapsto} \left(\begin{array}{c} 1 \\ -\lambda \\ -\frac{\lambda^3}{6} + \phi\theta\lambda \\ -\frac{\lambda^2}{2} + \phi\theta \\ \hline \theta \\ \phi \\ \phi\lambda \\ -\theta\lambda \end{array}\right),$$

with λ even parameter and θ, ϕ odd. By maximality, this supervariety $\mathcal{V} \subset \mathbb{P}(V)$ characterizes the reduction of the structure group $COSp(3|2) \subset CSpO(4|4)$.

Osculations of $\mathcal{V}$

Repeatedly applying $\mathfrak{f}_{-1}$ to $[x^3]$ yields the so-called osculating sequence

$$0 \subset V^0 \subset V^1 \subset V^2 \subset V^3 = V$$

of higher order affine tangent spaces of $\mathcal{V}$ at $[x^3]$.

Important fact I: The affine tangent space $V^1 \subset V \cong \mathbb{C}^{4|4}$ is Lagrangian w.r.t. $CSpO$-structure on V (in particular $\dim V^1 = (2|2)$).

Important fact II: The associated graded v.s. $\mathrm{gr}(V) = N_0 \oplus \cdots \oplus N_3$ has natural $\mathfrak{osp}(1|2)$-equivariant $\mathbb{Z}$-graded superalgebra structure and $N_1 \otimes N_1 \to N_2 \cong N_1^*$ is a supersymmetric cubic form $\mathfrak{C} \in S^3 N_1^*$ on $N_1 \cong \mathbb{C}^{1|2}$. (It dualizes to the product of simple Jordan superalgebra structure on N_1 called the Kaplansky superalgebra.) Explicitly $\mathfrak{C} = \frac{1}{3}\lambda^3 + 2\lambda\theta\phi$.

Formal framework for 2nd order super-PDE

Global	Local
Contact supermfld $(M^{5\vert 4}, \mathcal{C}) \cong J^1(\mathbb{C}^{2\vert 2}, \mathbb{C}^{1\vert 0})$	(x^i, u, u_i), $\sigma = du - \sum_{i=1}^4 u_i dx^i$ $\mathcal{C} = \langle \sigma = 0 \rangle = \langle \partial_{x^i} + u_i \partial_u, \partial_{u_i} \rangle$
$\mathcal{C}$ has frames of conformal symplectic-orthogonal supervector fields	$d\sigma\vert_{\mathcal{C}} = \left(\begin{array}{cccc\|cccc} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ \hline -1 & & & & & & & \\ & -1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \end{array} \right)$ $\partial_{x^i} + u_i \partial_u,\ \partial_{u_i}$ is adapted frame
Lagrangian subspace of $\mathcal{C}$ at $m \in M$	$\langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j} \rangle$
Lagrange–Grassmann bundle $(\widetilde{M}^{9\vert 8}, \widetilde{\mathcal{C}}) \cong J^2(\mathbb{C}^{2\vert 2}, \mathbb{C}^{1\vert 0})$	$(x^i, u, u_i, u_{ij} = \pm u_{ji})$ $\widetilde{\mathcal{C}} = \langle \partial_{x^i} + u_i \partial_u + u_{ij} \partial_{u_j}, \partial_{u_{ij}} \rangle$

A 2nd order super-PDE is a submanifold of Lagrange–Grassmann bundle $\widetilde{M}$ and an external symmetry is a symmetry of $(\widetilde{M}, \widetilde{\mathcal{C}})$ that preserves the submanifold.

Key steps of the proof

- Lagrangian lift. At any "point" of $(M, \mathcal{C})$ we have $(1|2)$-parametric family of Lagrangian subspaces of $\mathcal{C}$: the affine tangent spaces along $\mathcal{V}$. It gives $(6|6)$-dimensional submanifold $\mathcal{E} \subset \widetilde{M}$, i.e., the $G(3)$-contact super-PDE;
- Cubic form. The $G(3)$-contact super-PDE can be parametrically written as

$$(u_{ij}) = \begin{pmatrix} u_{00} & u_{0a} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(T^3) & \frac{3}{2}\mathfrak{C}_a(T^2) \\ \frac{3}{2}\mathfrak{C}_a(T^2) & 3\mathfrak{C}_{ab}(T) \end{pmatrix} \qquad (a, b = 1, 2, 3) \; .$$

  This extends to $G(3)$ a formula giving geometric realizations of exceptional Lie algebras – for different cubic forms – obtained by D. The in 2018.
- Symmetries. External symmetries of $G(3)$-contact super-PDE are derived explicitly by a hand computation using expression of generating functions on $(M, \mathcal{C})$ via the cubic form on the Kaplansky superalgebra;

Key steps of the proof

- Spencer cohomology. The previous computation tells that supersymmetry dimension is $(17|14)$, i.e., upper bound coming from Tanaka–Weisfeiler prolongation is attained. Moreover, $\exists$ grading element $\Longrightarrow$ the symmetry superalgebra is exactly $G(3)$.
- Cauchy characteristic reduction. On $\mathcal{E} \cong G(3)/P_{12}$ we have the Cartan superdistribution $\mathcal{H} \subset \mathcal{T}\mathcal{E}$ of rank $(3|4)$. The Cauchy characteristic space

$$\mathrm{Ch}(\mathcal{H}) = \{X \in \Gamma(\mathcal{H}) \mid \mathcal{L}_X \mathcal{H} \subset \mathcal{H}\}$$

  is a module for the space of superfunctions of $\mathcal{E}$ and it is generated by a nowhere-vanishing even supervector field. The quotient $\overline{\mathcal{E}} = \mathcal{E}/\,\mathrm{Ch}(\mathcal{H})$ is then $(5|6)$-dimensional and is endowed with superdistribution of rank $(2|4)$.
- SHC-equation. We have $\overline{\mathcal{E}} \cong G(3)/P_2$ with the Cartan superdistribution associated to SHC-eqn.

Geometric realizations of $F(4)$

Exceptional Lie superalgebra $F(4) = \big(\mathfrak{so}(7) \oplus \mathfrak{sp}(2)\big) \oplus \big(\mathbb{S} \otimes \mathbb{C}^2\big)$.

F(4) mixed-contact super-PDE:

$$(u_{ij}) = \begin{pmatrix} u_{00} & u_{0a} \\ u_{a0} & u_{ab} \end{pmatrix} = \begin{pmatrix} \mathfrak{C}(T^3) & \frac{3}{2}\mathfrak{C}_a(T^2) \\ \frac{3}{2}\mathfrak{C}_a(T^2) & 3\mathfrak{C}_{ab}(T) \end{pmatrix} \qquad (a, b = 1, 2, 3, 4)$$

where $u : \mathbb{C}^{3|2} \to \mathbb{C}^{1|0}$ and $\mathfrak{C} = \lambda\mu^2 + 2\mu\theta\phi$.

F(4) odd-contact super-PDE:

$$u_{0ab} = u_{ab}u_{123}, \quad 1 \leqslant a < b \leqslant 3.$$

where $u : \mathbb{C}^{0|4} \to \mathbb{C}^{1|0}$. This is a quite different story!

Thm[S., The] These super-PDE have symmetry superalgebras $F(4)$.

Thanks!