# Jet functors in noncommutative geometry

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# Noncommutative differential geometry

**Starting point:** study a geometric object via an algebra of "regular" functions over it (e.g. $\mathcal{C}^\infty(M)$, $\mathcal{O}(M)$, $\Bbbk[x_1, \ldots, x_n]/I$).
**Main idea:** the algebraic object becomes the focus of study, it is generalised and interpreted as the algebraic dual of a more general notion of space.

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| Geometry | Algebra | NCDG | Structure |
|---|---|---|---|
| $\mathbb{R}$ (or $\mathbb{C}$) | $\mathbb{R}$ (or $\mathbb{C}$) | $\Bbbk$ | comm. unital ring |
| $M$ | $\mathcal{C}^\infty(M)$ | $A$ | unital assoc. $\Bbbk$-algebra |
| $E$ v.b. | $\Gamma(E)$ | $E$ | f.g.p. left $A$-module |
| $E \to F$ | $\Gamma(E) \to \Gamma(F)$ | $E \to F$ | left $A$-linear map |

Useful categories: ${}_A\mathrm{FGP} \subseteq {}_A\mathrm{Proj} \subseteq {}_A\mathrm{Flat} \subseteq {}_A\mathrm{Mod}$, $\mathrm{Mod}_A$.

In order to describe the differential structure, we equip $A$ with a generalised notion of exterior algebra over it.

**Definition (Exterior algebra over $A$)**

*Associative graded algebra* $(\Omega_d^\bullet, \wedge)$ *with* $\Omega_d^0 = A$, *endowed with a* **differential**, *i.e. a* $\Bbbk$*-linear map* $d\colon \Omega_d^n \to \Omega_d^{n+1}$ *such that:*

- $d^2 = 0$;
- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^n \alpha \wedge (d\beta)$ *for* $\alpha \in \Omega_d^n$, $\beta \in \Omega_d^h$;
- $A$ *and* $dA$ *generate* $\Omega_d^\bullet$ *via* $\wedge$.

Examples:

- de Rham complex $(\Omega^\bullet(M), \wedge, d)$;
- *universal exterior algebra* $(\Omega_u^\bullet, d_u, \otimes_A)$.

In particular, for the first grade (universal first order diff. calculus):

$$\Omega^1_u = \ker\left(\cdot\colon A \otimes A \longrightarrow A\right)$$

with differential

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*Universal property*: given an exterior algebra $\Omega^\bullet_d$ on $A$, there exists a unique surjective map

$$\Omega^\bullet_u \longrightarrow\!\!\!\!\rightarrow \Omega^\bullet_d.$$

compatible with the algebra structure: grading, $d$, $\wedge$.
Explicitly, $\sum_i a_i \otimes b_i \in \Omega^1_u$ is mapped to $\sum_i a_i db_i \in \Omega^1_d$.

# Jet bundles

Given a vector bundle $E \to M$, the associated $n$-jet bundle $J^nE \to M$ represents the bundle of $n$-th order approximations of sections of $E$ (equivalence classes up to $n$-th order contact).

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Given a vector bundle $E \to M$, the associated $n$-jet bundle $J^n E \to M$ represents the bundle of $n$-th order approximations of sections of $E$ (equivalence classes up to $n$-th order contact). They provide:

- an intrinsic notion of "Taylor approximation";
- a characterisation of differential operators;
- an intrinsic definition of differential equation;
- a tool for a theory of differential equations.

Jet bundles come equipped with:

- $n$-**jet prolongation** ($\mathbb{R}$-linear) map $\forall n \geq 0$

$$j^n \colon \Gamma(E) \hookrightarrow \Gamma(J^n E), \qquad \sigma \longmapsto [\sigma]^n;$$

- **jet projections** (vector bundle maps) $\forall n \geq m \geq 0$

$$\pi^{n,m} \colon J^n E \twoheadrightarrow J^m E, \qquad [\sigma]_p^n \longmapsto [\sigma]_p^m.$$

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This construction is functorial: $\phi \colon E \to F$ gives

$$J^n\phi \colon J^n E \longrightarrow J^n F, \qquad [\sigma]^n_p \longmapsto [\phi \circ \sigma]^n_p,$$

and the following are natural transformations

$$j^n \colon \mathrm{id}_{{}_A\mathrm{Mod}} \longrightarrow \Gamma \circ J^n \qquad \pi^{n,m} \colon J^n \longrightarrow J^m,$$

such that

$$\pi^{n,m} \circ \pi^{m,h} = \pi^{n,h}, \qquad \pi^{n,m} \circ j^n = j^m.$$

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Taking global sections we obtain the following short exact sequence of finitely generated projective $\mathcal{C}^\infty(M)$-modules (equivalent by Serre-Swann)

$$0 \longrightarrow \Gamma(S^n E) \longrightarrow \Gamma(J^n E) \xrightarrow{\Gamma\pi^{n,n-1}} \Gamma(J^{n-1} E) \longrightarrow 0.$$

Given an exterior algebra $\Omega_d^\bullet$ over $A$, we need to find functors $S_d^n, J_d^n \colon {}_A\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod}$ and natural transformations

$$\iota_d^n \colon S_d^n \longrightarrow J_d^n, \qquad \pi_d^{n,m} \colon J_d^n \longrightarrow J_d^m,$$

fitting in the following short exact sequence

$$0 \longrightarrow S_d^n \xhookrightarrow{\iota_d^n} J_d^n \xrightarrow{\pi_d^{n,n-1}} J_d^{n-1} \longrightarrow 0.$$

Furthermore, we want a $\Bbbk$-linear natural transformation $j_d^n \colon \mathrm{id}_{{}_A\mathrm{Mod}} \longrightarrow J_d^n$ such that

$$\pi_d^{n,m} \circ \pi_d^{m,h} = \pi_d^{n,h}, \qquad \pi_d^{n,m} \circ j_d^n = j_d^m.$$

# Quantum symmetric forms

In the classical case, the $\mathcal{C}^\infty(M)$-module of differential forms with values in a bundle $E$ can be seen as $\Omega^\bullet(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E)$.
So, given an exterior algebra $\Omega_d^\bullet$ over $A$, we can define the functors

$$\Omega_d^\bullet \colon {}_A\mathrm{Mod} \longrightarrow {}_A\mathrm{Mod} \qquad E \longmapsto \Omega_d^\bullet \otimes_A E;$$
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$$S_d^0 = \Omega_d^0 = \mathrm{id}_{{}_A\mathrm{Mod}}, \qquad S_d^1 = \Omega_d^1 := \Omega_d^1 \otimes_A -.$$

For $n > 1$, the **functor of quantum symmetric forms** $S_d^n$ is defined by induction as the kernel of the following composition

$$\Omega_d^1 \circ S_d^{n-1} \xrightarrow{\Omega_d^1(\iota_\wedge^{n-1})} \Omega_d^1 \circ \Omega_d^1 \circ S_d^{n-2} \xrightarrow{\wedge_{S_d^{n-2}}} \Omega_d^2 \circ S_d^{n-2}$$

and $\iota_\wedge^n\colon S_d^n \longrightarrow \Omega_d^1 \circ S_d^{n-1}$ is the inclusion.

The following lemma shows other equivalent descriptions of $S^n_d$.

## Lemma 1

*If $\Omega^1_d$ and $\Omega^2_d$ are flat in $\mathrm{Mod}_A$, for all $n \geq 0$, the following subfunctors of the tensor algebra $T^n_d := (\Omega^1_d)^{\otimes_A n}$ coincide*

1. $S^n_d$;
2. $\bigcap_{k=0}^{n-2} \ker\left(T^k_d \wedge_{T^{n-k-2}_d}\right)$;
3. $\bigcap_{\substack{h\geq 2\\ 0\leq k\leq n-h}} \ker\left(T^k_d(\wedge_h)_{T^{n-k-h}_d}\right)$, *where* $\wedge_h \colon T^h_d \longrightarrow \Omega^h_d$;
4. $\left(S^h_d \circ T^{n-h}_d\right) \cap \left(T^{n-k}_d \circ S^k_d\right)$ *for* $0 \leq h, k \leq n$ *such that* $h + k > n$.

# Spencer cohomology

For all $k, h \geq 0$, consider the functor $\Omega_d^k \circ S_d^h$, and define $\delta^{h,k}$ as

$$\Omega_d^k \circ S_d^h \xrightarrow{\Omega_d^k(\iota_\wedge^h)} \Omega_d^k \circ \Omega_d^1 \circ S_d^{h-1} \xrightarrow{(-1)^k \wedge^{k,1}_{S_d^{h-1}}} \Omega_d^{k+1} \circ S_d^{h-1}$$

$$\delta^{h,k} : \Omega_d^k \circ S_d^h \longrightarrow \Omega_d^{k+1} \circ S_d^{h-1}$$

We get a complex in the category of functors of type ${}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$.

$$0 \longrightarrow S_d^n \xrightarrow{\delta^{n,0}} \Omega_d^1 \circ S_d^{n-1} \xrightarrow{\delta^{n-1,1}} \Omega_d^2 \circ S_d^{n-2} \xrightarrow{\delta^{n-2,2}} \Omega_d^3 \circ S_d^{n-3} \cdots$$

**Definition (Spencer cohomology)**

*We call this the Spencer $\delta$-complex, its cohomology the Spencer cohomology, and we denote the cohomology at $\Omega_d^k \circ S_d^h$ by $H^{h,k}$.*

# Universal 1-jet module

We start from the simplest case by computing $J^1_u E$ for $E = A$ (classically $A = \mathcal{C}^\infty(M) \cong \Gamma(M \times \mathbb{R})$).

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Since $S^1_u = \Omega^1_u = \ker(\cdot) \subset A \otimes A$, we have a natural way of building the 1-jet short exact sequence, that is

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We thus define $J^1_u A := A \otimes A$ (free 1-dim. $A$-bimodule), where the projection $\pi^{1,0}_u \colon J^1_u A \longrightarrow A$ is the algebra multiplication.
We take as universal prolongation $j^1_u \colon a \mapsto 1 \otimes a$, which splits the sequence in $\mathrm{Mod}_A$.

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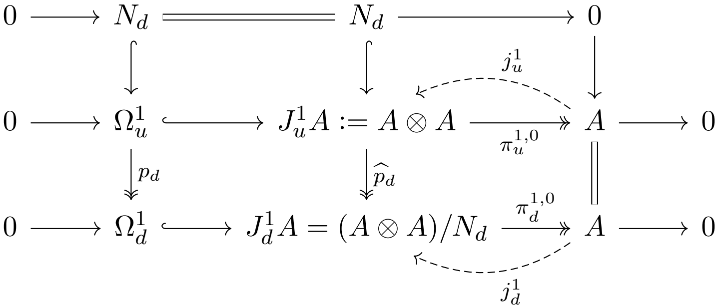


We thus define $J^1_d A := A \otimes A/N_d$,

$$\pi^{1,0}_d([a \otimes b]) := ab, \qquad\qquad j^1_d(a) := [1 \otimes a].$$

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In order to obtain the short exact sequence for all $E$ in ${}_A\mathrm{Mod}$ we can apply the functor $- \otimes_A E \colon {}_A\mathrm{Mod} \longrightarrow \mathrm{Mod}$.
The 1-jet sequence splits in $\mathrm{Mod}_A$, so it remains exact.

# Nonholonomic jet functors

## Definition

*We term the functor*

$$J_d^{(n)} := (J_d^1)^{\circ n} = J_d^1 \circ \cdots \circ J_d^1 = (J_d^1 A)^{\otimes_A n} \otimes_A - \colon {}_A\mathrm{Mod} \to {}_A\mathrm{Mod}$$

*the* **nonholonomic $n$-jet functor**. *The following composition is called the* **nonholonomic $n$-jet prolongation**.

$$j_d^{(n)} := j^1_{d,J_d^{(n)}} \circ j^1_{d,J_d^{(n-1)}} \circ \cdots \circ j^1_{d,J_d^1} \circ j_d^1 \colon \mathrm{id} \longrightarrow J_d^{(n)}.$$

*For all $1 \leq m \leq n$, we have the natural epimorphisms*

$$\pi_d^{(n,n-1;m)} = J_d^{(n-m)} \pi^{1,0}_{d,J_d^{(m-1)}} \colon J_d^{(n)} \twoheadrightarrow J_d^{(n-1)},$$

*which will be called the* **nonholonomic $n$-jet projections**.

# 2-jet functor

We build the (holonomic) $2$-jet module with the aim that the following sequence is exact

$$0 \longrightarrow S^2_d E \xhookrightarrow{\iota^2_{d,E}} J^2_d E \xrightarrow{\pi^{2,1}_{d,E}} J^1_d E \longrightarrow 0$$

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$$0 \longrightarrow \Omega^1_d(J^1_d E) \xhookrightarrow[\iota^1_{d,J^1_d E}]{} J^{(2)}_d E \xrightarrow[\pi^{1,0}_{d,J^1_d E}]{} J^1_d E \longrightarrow 0$$

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We assume $\Omega^1_d$ flat in $\mathrm{Mod}_A$. As for the classical case, we expect the jet prolongation to agree with the nonholonomic one, i.e.

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Under these conditions, $j^{(2)}_d(E) + S^2_d E \subseteq J^{(2)}_d E$ satisfies the 2-jet short exact sequence.

We can describe $J_d^2 E$ implicitly as the kernel of a bilinear map

$$\widetilde{\text{Đ}}_E \colon J_d^{(2)} E \longrightarrow (\Omega_d^1 \ltimes \Omega_d^2)(E),$$

where $(\Omega_d^1 \ltimes \Omega_d^2)(E) \cong (\Omega_d^1 \ltimes \Omega_d^2) \otimes_A E$.
As a right $A$-module, $\Omega_d^1 \ltimes \Omega_d^2 \cong \Omega_d^1 \oplus \Omega_d^2$, but as an $A$-bimodule, it comes equipped with a non-trivial left action

$$f \star (\alpha + \omega) = f\alpha + df \wedge \alpha + f\omega, \quad \forall f \in A,\ \alpha \in \Omega_d^1,\ \omega \in \Omega_d^2.$$

Explicitly, we have

$$\begin{aligned} \widetilde{\text{Đ}}_E \colon J_d^{(2)} E &\longrightarrow (\Omega_d^1 \ltimes \Omega_d^2)(E) \\ [a \otimes b] \otimes_A [c \otimes e] &\longmapsto (ad(bc) \otimes_A e, da \wedge d(bc) \otimes_A e)\,. \end{aligned}$$

**Definition (Holonomic $n$-jet functor)**

*Let $A$ be a $\Bbbk$-algebra endowed with an exterior algebra $\Omega^\bullet_d$ over it. We define $J^n_d$ as the kernel of the natural transformation*

$$J^1_d \circ J^{n-1}_d \xrightarrow{J^1_d(l^{n-1}_d)} J^1_d \circ J^1_d \circ J^{n-2}_d \xrightarrow{\widetilde{\text{Đ}}_{J^{n-2}_d}} (\Omega^1_d \ltimes \Omega^2_d) \circ J^{n-2}_d,$$

*where we denote the natural inclusion by $l^n_d \colon J^n_d \longrightarrow J^1_d \circ J^{n-1}_d$. We call $J^n_d$ the* **(holonomic) $n$-jet functor**.

It is natural to consider the following composition

$$\iota_{J^n_d} := J^{(n-2)}_d(l^2_d) \circ J^{(n-3)}_d(l^3_d) \circ \cdots \circ J^{(1)}_d(l^{n-1}_d) \circ l^n_d \colon \mathbf{J^n_d} \longrightarrow \mathbf{J^{(n)}_d}.$$

In general, $\iota_{J^n_d}$ is not injective (as has been noted also in the setting of synthetic differential geometry).

We define the **(holonomic)** $n$**-jet projection** as the natural transformation $\pi^{n,n-1}$ obtained as the composition

$$J_d^n \xhookrightarrow{l_d^n} J_d^1 \circ J_d^{n-1} \xrightarrow{\pi^{1,0}_{d,J_d^{n-1}}} J_d^{n-1}.$$

More generally, by composing them, we get, for all $0 \leq m \leq n$,

$$\pi_d^{n,m} := \pi_d^{m+1,m} \circ \pi_d^{m+2,m+1} \circ \cdots \circ \pi_d^{n,n-1} \colon \mathbf{J_d^n} \longrightarrow \mathbf{J_d^m}.$$

The natural map $\iota_d^n$ is defined by induction, for $n \geq 2$ as the unique morphism that commutes in the following diagram

$$\begin{array}{ccccc}
S_d^n & \xhookrightarrow{\iota_\wedge^n} & \Omega_d^1 \circ S_d^{n-1} & \xrightarrow{\Omega_d^1(\iota_d^{n-1})} & \Omega_d^1 \circ J_d^{n-1} \\
{\scriptstyle \iota_d^n} \downarrow & & & & \downarrow {\scriptstyle \iota^1_{d,J_d^{n-1}}} \\
J_d^n & & \xhookrightarrow{\qquad l_d^n \qquad} & & J_d^1 \circ J_d^{n-1}
\end{array}$$

## Theorem (Holonomic jet exact sequence)

*Let $A$ be a $\Bbbk$-algebra endowed with an exterior algebra $\Omega^\bullet_d$ such that $\Omega^1_d$, $\Omega^2_d$, and $\Omega^3_d$ are flat in $\mathrm{Mod}_A$. For $n \geq 1$, if the Spencer cohomology $H^{m,2}$ vanishes, for all $1 \leq m < n-2$, then the following sequence is exact,*

$$0 \longrightarrow S^n_d \xhookrightarrow{\iota^n_d} J^n_d \xrightarrow{\pi^{n,n-1}_d} J^{n-1}_d \longrightarrow H^{n-2,2}.$$

*Therefore, if $H^{n-2,2} = 0$ we obtain a short exact sequence*

$$0 \longrightarrow S^n_d \xhookrightarrow{\iota^n_d} J^n_d \xrightarrow{\pi^{n,n-1}_d} J^{n-1}_d \longrightarrow 0.$$

**Theorem (Stability)**

*Let $A$ be a $\Bbbk$-algebra endowed with an exterior algebra $\Omega_d^\bullet$.*

1. *If $\Omega_d^1$ is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), then $J_d^{(n)}$* **preserves** ${}_A\mathrm{Flat}$ **(resp.** ${}_A\mathrm{Proj}$**,** ${}_A\mathrm{FGP}$**)***;*
2. *If $\Omega_d^1$, $\Omega_d^2$, and $\Omega_d^3$ are flat in $\mathrm{Mod}_A$, $H^{m,2}$ vanishes and $S_d^m$ is in ${}_A\mathrm{Flat}$ (resp. ${}_A\mathrm{Proj}$, ${}_A\mathrm{FGP}$), for all $1 \leq m \leq n$, then $J_d^n$* **preserves** ${}_A\mathrm{Flat}$ **(resp.** ${}_A\mathrm{Proj}$**,** ${}_A\mathrm{FGP}$**)**.

These functors are reasonable, as they map bundles into bundles.

**Theorem (Classical correspondence)**

*Let $A = C^\infty(M)$ for a smooth manifold $M$, let $\Omega^\bullet_d = \Omega^\bullet(M)$ equipped with the de Rham differential $d$, and let $E$ be the space of smooth sections of a vector bundle. Then the $\mathcal{C}^\infty(M)$-modules of sections of the associated classical nonholonomic and holonomic $n$-jet bundles are isomorphic to $J^{(n)}_d E$ and $J^n_d E$ in ${}_A\mathrm{Mod}$, respectively, and the prolongation maps and jet projections are compatible with the isomorphisms.*

## Definition (Differential operators)

*Let $E, F \in {}_A\mathrm{Mod}$. A $\Bbbk$-linear map $\Delta\colon E \to F$ is called a* linear differential operator *of order at most $n$ with respect to the exterior algebra $\Omega^\bullet_d$, if there exists an* $A$-module map *$\widetilde{\Delta} \in {}_A\mathrm{Hom}(J^n_d E, F)$ such that the following diagram commutes:*

$$\begin{array}{ccc} J^n_d E & & \\ {\scriptstyle j^n_d}\uparrow & \overset{\widetilde{\Delta}}{\searrow} & \\ E & \xrightarrow{\ \Delta\ } & F \end{array}$$

*If $n$ is* minimal*, we say that $\Delta$ is of* order *$n$.*

- stability under sum and composition;
- what should be a differential operator is a differential operator (connections, $d$, partial derivatives, $\text{Đ}_E = \widetilde{\text{Đ}}_E \circ j^1_{J^1_d E}$);
- new tool to build exterior algebras (terminal calculi).

# Thank you!