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Atiyah sequences of braided Lie algebras and their splittings

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This is a study on the gauge group of NC principal bundles.
We study infinitesimal gauge transformations.

A (NC) principal bundle is seen as a Hopf-Galois extension $B = A^{coH} \subseteq A$, where

H=Hopf algebra

A= total space algebra, in a right H-comodule algebra

B=base space algebra, subalgebra of H-coinvariant elements of A.

Hopf-Galois extension property, bijectivity of the canonical map:

$$\chi : A \otimes_B A \longrightarrow A \otimes H \ , \quad a' \otimes_B a \longmapsto a' a_{(0)} \otimes a_{(1)}$$

A first def. of infinitesimal gauge transformations:

Derivations $\psi : A \to A$ that are H-equivariant and are trivial on B.

H-equivariance: $\delta(\phi(b)) = \psi(b_{(0)}) \otimes b_{(1)}$.

Gauge transf. form a Lie algebra under usual commutator bracket. Similarly, finite Gauge transformations for a group.

This definition is very restrictive since there are few derivations in a NC algebra A.

Example: $\mathbb{C} \subseteq H$ is a Hopf–Galois extension (Galois object, $A = H$, $B = \mathbb{C}$).

Group of finite gauge gransformations $\simeq$ characters of H.

We study Hopf–Galois etensions $A^{coH} \subseteq A$ that are K-equivariant where K is a triangular Hopf algebra:

K is trivially a K-module algebra (trivial action of K on H).

A is a K-module algebra.

The H-coaction and the K-action on A commute.

In this setting we can study the Hopf–Galois extension $B \subset A$ as an object internal to the category of K-modules.

A Hopf algebra in the category of K-modules is a K-braided Hopf algebra [Majid]; a Lie algebra is a K-braided Lie algebra.

Def. A K-braided Lie algebra g is

a K-module g

with $[\ ,\ ] : \mathrm{g} \otimes \mathrm{g} \to \mathrm{g}$ such that

(i) K-equivariance: for $\Delta(k) = k_{(1)} \otimes k_{(2)}$ the coproduct of K,

$$k \triangleright [u, v] = [k_{(1)} \triangleright u, k_{(2)} \triangleright v]$$

(ii) braided antisymmetry:mmetry:

$$[u, v] = -[\mathsf{R}_\alpha \triangleright v, \mathsf{R}^\alpha \triangleright u],$$

(iii) braided Jacobi identity:

$$[u, [v, w]] = [[u, v], w] + [\mathsf{R}_\alpha \triangleright v, [\mathsf{R}^\alpha \triangleright u, w]],$$

$u, v, w \in \mathrm{g}$, $k \in K$,

$$\mathsf{R} = \mathsf{R}^\alpha \otimes \mathsf{R}_\alpha \in K \otimes K$$

triangular structure of K (universal R-matrix).

Example: since A is a K-module *Hom*(A,A) is naturally a K-module with action

$$\triangleright_{Hom(A,A)}\colon K\otimes Hom(A,A)\to Hom(A,A)$$
$$k\otimes\psi\mapsto k\triangleright_{Hom(A,A)}\psi:\ A\mapsto k_{(1)}\triangleright_A\psi(S(k_{(2)})\triangleright A)$$

Hom(A,A) is a K-braided Lie algebra with bracket

$$[\ ,\ ]: Hom(A,A)\otimes Hom(A,A)\to Hom(A,A),$$
$$\psi\otimes\psi'\ \mapsto[\psi,\psi']=\psi\psi'-(\mathsf{R}_\alpha\triangleright\psi')\circ(\mathsf{R}^\alpha\triangleright\psi)\,.$$

Elements ψ in *Hom*(A,A) which satisfy

$$\psi(aa')=\psi(a)a'+(\mathsf{R}_\alpha\triangleright a)\,(\mathsf{R}^\alpha\triangleright_{Hom(A,A)}\psi)(a') \qquad (1)$$

on the product of two elements a,a' of A are called **braided derivations.**

$$\mathrm{Der}(A):=\{\psi\in Hom(A,A)\,|\,$$
$$\psi(aa')=\psi(a)a'+(\mathsf{R}_\alpha\triangleright a)\,(\mathsf{R}^\alpha\triangleright_{Hom(A,A)}\psi)(a')\}$$

the $\mathbb{K}$-module of braided derivations of A. It is a **K-braided Lie subalgebra of** *Hom*(A,A), with

$$[\ ,\ ]:\mathrm{Der}(A)\otimes\mathrm{Der}(A)\to\mathrm{Der}(A)$$
$$\psi\otimes\lambda\mapsto[\psi,\lambda]:=\psi\circ\lambda-(\mathsf{R}_\alpha\triangleright_{\mathrm{Der}(A)}\lambda)\circ(\mathsf{R}^\alpha\triangleright_{\mathrm{Der}(A)}\psi).$$

Let the K-module algebra A be *quasi-commutative*:

$$a\, a' = (\mathsf{R}_\alpha \triangleright a')\, (\mathsf{R}^\alpha \triangleright a) \ ,$$

$\mathsf{Der}(A)$ is an A-module with $A \otimes \mathsf{Der}(A) \to \mathsf{Der}(A)$,

$$(a\psi)(a') := a\, \psi(a'), \tag{2}$$

for $\psi \in$ *Hom*(A, A), $a, a' \in A$, is also a left A-submodule of *Hom*(A, A). Furthermore:

$$[\psi, a\psi'] = \psi(a)\psi' + (\mathsf{R}_\alpha \triangleright a)[\mathsf{R}^\alpha \triangleright \psi, \psi'] \tag{3}$$

for all $a, a' \in A, \psi, \psi' \in \mathsf{Der}(A)$.

For A K-braided commutative, $(A,\ \mathsf{Der}(A))$ is a K**-braided Lie-Rinehard pair**: A K-braided Lie algebra that satisfies (1), (2), (3).

Hopf-Galois

$B \subset A$ is K-equivariant Hopf-Galois extension, i.e., $B \subset A$ is a Hopf-Galois extension in the category of K-modules.

Definition . Let (K, R) be a triangular Hopf algebra. Infinitesimal gauge transformations of a K-equivariant Hopf–Galois extension $B = A^{coH} \subseteq A$ are H-comodule maps that are braided vertical derivations

$$\begin{aligned} \mathsf{aut}_B^{\mathsf{R}}(A) := \{u \in \mathit{Hom}(A, A) \mid \delta(u(a)) &= u(a_{(0)}) \otimes a_{(1)}, \\ u(aa') &= u(a)a' + (\mathsf{R}_\alpha \rhd a)(\mathsf{R}^\alpha \rhd u)(a'), \\ u(b) &= 0, \text{ for all } a, a' \in A, b \in B\} . \end{aligned}$$

Proposition. . *The linear space* $\mathsf{aut}_B^{\mathsf{R}}(A)$ *with bracket*

$$\begin{aligned} [\ ,\ ]_{\mathsf{R}} : \mathsf{aut}_B^{\mathsf{R}}(A) \otimes \mathsf{aut}_B^{\mathsf{R}}(A) &\to \mathsf{aut}_B^{\mathsf{R}}(A) \\ u \otimes u' &\mapsto [u, u']_{\mathsf{R}} := u \circ u' - \mathsf{R}_\alpha \rhd u' \circ \mathsf{R}^\alpha \rhd u , \end{aligned} \tag{4}$$

for all $u, u' \in \mathsf{aut}_B^{\mathsf{R}}(A)$*, is a* K*-braided Lie subalgebra of* $\mathsf{Der}^{\mathsf{R}}(A)$*.*

Example (quantum homogenous space).
Let (A, R) *be a cotriangular Hopf algebra with dual triangular Hopf algebra* $(U, \mathscr{R})$*. Let* $\mathfrak{R}$ *the triangular structure of* $U^{op} \otimes U$*.*

Let $B = A^{coH} \subseteq A$ *be a quantum principal bundle over the quantum homogeneous space* B*, with Hopf algebra projection* $\pi : A \to H$*. Assume* (H, R_H) *is cotriangular with* $R = R_H \circ (\pi \otimes \pi)$*.*

The braided gauge transformations $\mathrm{aut}_B^{\mathsf{R}}(A)$ *are the vertical braided vector fields in* $B \otimes \mathrm{Der}^{\mathfrak{R}}(A)_{\mathrm{inv}}$*, where* $\mathrm{Der}^{\mathfrak{R}}(A)_{\mathrm{inv}}$ *are the right-invariant vector fields defining the bicovariant differential calculus on* (A, R)*. This linear space isomorphism is a* $K = U_H^{op}$*-braided Lie algebra isomorphism, where* $U_H \subset U$ *is the triangular Hopf algebra dual to* H*.*

Examples form twist deformation (Twisting of braided Lie algebras).

Drinfeld twists $\mathsf{F} \in K \otimes K$ satisfies

$$(\mathsf{F} \otimes 1)[(\Delta \otimes \mathbf{I})(\mathsf{F})] = (1 \otimes \mathsf{F})[(\mathbf{I} \otimes \Delta)(\mathsf{F})] \ .$$

Notation: $\mathsf{F} = \mathsf{F}^{\alpha} \otimes \mathsf{F}_{\alpha}$ and $\overline{\mathsf{F}} := F^{-1} =: \overline{\mathsf{F}}^{\alpha} \otimes \overline{\mathsf{F}}_{\alpha}$ (implicit sum)

Prop. Let $\mathsf{F} = \mathsf{F}^{\alpha} \otimes \mathsf{F}_{\alpha}$ be a twist on $(K, m, \eta, \Delta, \varepsilon, S)$. Then the algebra K_{F} new Hopf algebra with coproduct

$$\Delta_{\mathsf{F}}(k) := \mathsf{F}\Delta(k)\overline{\mathsf{F}} = \mathsf{F}^{\alpha}k_{(1)}\overline{\mathsf{F}}^{\beta} \otimes \mathsf{F}_{\alpha}k_{(2)}\overline{\mathsf{F}}_{\beta} \ , \qquad k \in K \tag{5}$$

and antipode S_{F}. If (K, R) is (quasi)triangular $(K_{\mathsf{F}}, \mathsf{R}_{\mathsf{F}})$ is (quas)itirangular

$$\mathsf{R}_{\mathsf{F}} := \mathsf{F}_{21}\,\mathsf{R}\,\overline{\mathsf{F}} = \mathsf{F}_{\alpha}\mathsf{R}^{\beta}\overline{\mathsf{F}}^{\gamma} \otimes \mathsf{F}^{\alpha}\mathsf{R}_{\beta}\overline{\mathsf{F}}_{\gamma}$$

If A is a K-module algebra,A_{F} is a K_{F}-module algebra with

$$m_{A_{\mathsf{F}}} : A_{\mathsf{F}} \otimes_{\mathsf{F}} A_{\mathsf{F}} \longrightarrow A_{\mathsf{F}}, \qquad a \otimes_{\mathsf{F}} a' \longmapsto a \bullet_{\mathsf{F}} a' := (\overline{\mathsf{F}}^{\alpha} \triangleright_A a)\,(\overline{\mathsf{F}}_{\alpha} \triangleright_A a'). \tag{6}$$

When $\mathfrak{g}$ is a braided Lie algebra associated with a triangular Hopf algebra (K, R), and F is a twist for K, the K_{F}-module $\mathfrak{g}_F$ inherits from $\mathfrak{g}$ a twisted bracket

Prop The K_F-module g_F with bilinear map

$$[\ ,\ ]_\mathsf{F} = \mathsf{g}_\mathsf{F} \otimes \mathsf{g}_\mathsf{F} \to \mathsf{g}_\mathsf{F}\ ,\quad u \otimes v \to [u,v]_\mathsf{F} := [\overline{\mathsf{F}}^\alpha \triangleright u, \overline{\mathsf{F}}_\alpha \triangleright v]\ . \tag{7}$$

is a braided Lie algebra associated with $(K_\mathsf{F}, \mathsf{R}_\mathsf{F})$.

In particular we obtain the braided Lie algebra $(\mathsf{Der}(A)_\mathsf{F}), [\ ,\ ]_\mathsf{F})$ associated with $(K_\mathsf{F}, \mathsf{R}_\mathsf{F})$. The K_F-action $\triangleright_{\mathsf{Der}(A)_F}$ coincides with $\triangleright_{\mathsf{Der}(A)}$ as linear map.

Lie bracket:

$$[\psi,\lambda]_\mathsf{F} = \psi \circ_\mathsf{F} \lambda - (\mathsf{R}_{\mathsf{F}\alpha} \triangleright_{\mathsf{Der}(A)} \lambda) \circ_\mathsf{F} (\mathsf{R}_\mathsf{F}{}^\alpha \triangleright_{\mathsf{Der}(A)} \psi)\ ,$$

with the composition that is changed as in (6):

$$\psi \circ_\mathsf{F} \phi = (\overline{\mathsf{F}}^\alpha \triangleright_{\mathsf{Der}(A)} \psi) \circ (\overline{\mathsf{F}}_\alpha \triangleright_{\mathsf{Der}(A)} \phi)\ .$$

On the other hand, braided Lie algebra $\mathsf{Der}(A_\mathsf{F})$ of the K_F-module A_F associated with $(K_\mathsf{F}, \mathsf{R}_\mathsf{F})$. The K_F-action is

$$\triangleright_{\mathsf{Der}(A_\mathsf{F})} \colon K_\mathsf{F} \otimes \mathsf{Der}(A_\mathsf{F}) \to \mathsf{Der}(A_\mathsf{F}) k \otimes \psi \mapsto k \triangleright_{\mathsf{Der}(A_\mathsf{F})} \psi :\ a \mapsto h_{\widetilde{(1)}} \triangleright_{A_\mathsf{F}} \psi($$

with bracket

$$[\psi,\lambda]_{\mathsf{R}_\mathsf{F}} = \psi \circ \lambda - (\mathsf{R}_{\mathsf{F}\alpha} \triangleright_{\mathsf{Der}(A_\mathsf{F})} \lambda) \circ (\mathsf{R}_\mathsf{F}{}^\alpha \triangleright_{\mathsf{Der}(A_\mathsf{F})} \psi)\ .$$

These two braided Lie algebras are isomorphic

Theorem. . *The braided Lie algebras* $(\mathsf{Der}(A)_\mathsf{F}, [\ ,\ ]_\mathsf{F})$ *and* $(\mathsf{Der}(A_\mathsf{F}), [\ ,\ ]_{\mathsf{R}_\mathsf{F}})$ *are isomorphic via the map*

$$\mathcal{D} : \mathsf{Der}(A)_\mathsf{F} \to \mathsf{Der}(A_\mathsf{F}), \qquad \psi \mapsto \mathcal{D}(\psi) : a \mapsto (\overline{\mathsf{F}}^\alpha \triangleright_{\mathsf{Der}(A)_\mathsf{F}} \psi)(\overline{\mathsf{F}}_\alpha \triangleright_A a)\ , \tag{8}$$

which satisfies $\mathcal{D}\Big([\psi, \lambda]_\mathsf{F}\Big) = [\mathcal{D}(\psi), \mathcal{D}(\lambda)]_{\mathsf{R}_\mathsf{F}}$*, for all* $\psi, \lambda \in \mathsf{Der}(A)_\mathsf{F}$*.*

Remark: The twist gives and isomorphism of braided monoidal caltegories and, using $\mathcal{D}$*, of closed braided monoidal categories.*

Isomorphism of the Lie-Rinehard structures.

For $\mathsf{Der}(A)_\mathsf{F}$*:*

$$a \cdot_\mathsf{F} \psi := (\overline{\mathsf{F}}^\alpha \triangleright_A a)\, (\overline{\mathsf{F}}_\alpha \triangleright_{\mathsf{Der}(A)} \psi)\, , \tag{9}$$

$$[\psi, a \cdot_\mathsf{F} \psi']_\mathsf{F} = [\psi, a]_\mathsf{F} \cdot_\mathsf{F} \psi' + ({\mathsf{R}_\mathsf{F}}_\alpha \triangleright a) \cdot_\mathsf{F} [{\mathsf{R}_\mathsf{F}}^\alpha \triangleright \psi, \psi']_\mathsf{F} \tag{10}$$

where $[\psi, a]_\mathsf{F} = [\overline{\mathsf{F}}^\alpha \triangleright \psi, \overline{\mathsf{F}}_\alpha \triangleright a] = (\overline{\mathsf{F}}^\alpha \triangleright \psi)(\overline{\mathsf{F}}_\alpha \triangleright a)$*.*

For $\mathsf{Der}(A_\mathsf{F})$*:*

$$(a \bullet_\mathsf{F} \psi)(a') := a \bullet_\mathsf{F} \psi(a') \tag{11}$$

$$[\psi, a \bullet_{\mathsf{F}} \psi']_{\mathsf{F}} = [\psi, a]_{\bullet_{\mathsf{F}}} \psi' + (\mathsf{R}_{\mathsf{F}\alpha} \triangleright a) \cdot_{\mathsf{F}} [\mathsf{R}_{\mathsf{F}}{}^{\alpha} \triangleright \psi, \psi']_{\mathsf{F}} \tag{12}$$

for any $a, a' \in A_{\mathsf{F}}, \psi \in \mathsf{Der}(A_{\mathsf{F}})$*.*

The isomorphism $\mathcal{D} : \mathsf{Der}(A)_{\mathsf{F}} \to \mathsf{Der}(A_{\mathsf{F}})$ *respects the* A_{F}*-module structures and the compatibility of the* A_F *action with the brakets.*

$$\mathcal{D}(a \cdot_{\mathsf{F}} \psi) = a \bullet_{\mathsf{F}} \mathcal{D}(\psi) \ ,$$

for $a \in A_{\mathsf{F}}$ *and* $\psi \in \mathsf{Der}(A)_{\mathsf{F}}$*.*

Proposition. . *The isomorphism* $\mathcal{D} : (\mathsf{Der}(A)_{\mathsf{F}}, [\ ,\ ]_{\mathsf{F}}) \to (\mathsf{Der}(A_{\mathsf{F}}), [\ ,\ ]_{\mathsf{R}_{\mathsf{F}}})$ *of braided Lie algebras restricts to isomorphisms*

$$\mathcal{D} : \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)_{\mathsf{F}} \to \mathsf{Der}^{\mathsf{R}_{\mathsf{F}}}_{\mathcal{M}^{H_{\mathsf{F}}}}(A_{\mathsf{F}})$$

and

$$\mathcal{D} : \mathsf{aut}^{\mathsf{R}}_{B}(A)_{\mathsf{F}} \to \mathsf{aut}^{\mathsf{R}_F}_{B_{\mathsf{F}}}(A_{\mathsf{F}})$$

of $(K_{\mathsf{F}}, \mathsf{R}_{\mathsf{F}})$*-braided Lie algebras.*

where $\mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$ *is the* K*-submodule of* H*-equivariant derivations.*

Instanton bundle over S^4_θ and its gauge transformations

The $H = \mathcal{O}(SU(2))$ Hopf–Galois extension $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$

can be obtained as a twist on $K = \mathcal{O}(\mathbb{T}^2)$ of the Hopf–Galois extension $\mathcal{O}(S^4) \subset \mathcal{O}(S^7)$ of the classical $SU(2)$ Hopf bundle.

Study the classical Lie algebras

$$\mathrm{Der}(\mathcal{O}(S^7))\ ,\ \ \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)\ ,\ \ \mathrm{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$$

and twist to obtain the braided Lie algebras

$$\mathrm{Der}(\mathcal{O}(S^7_\theta))\ ,\ \ \mathrm{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))\ ,\ \ \mathrm{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$$

Classical $SU(2)$*-Hopf bundle* $\pi : S^7 \to S^4$.

$A := \mathcal{O}(S^7)$ *is* $*$*-algebra of coordinate functions on the* S^7

generators: $\{z_a, z_a^*, a = 1, \ldots, 4\}$, *sphere relation* $\sum z_a^* z_a = 1$. *Coaction:*

$$\delta : \quad \mathcal{O}(S^7) \longrightarrow \mathcal{O}(S^7) \otimes \mathcal{O}(SU(2)) \tag{13}$$

$$\mathsf{u} \longmapsto \mathsf{u} \dot{\otimes} \mathsf{w}\,, \quad \mathsf{u} := \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2^* & z_1^* & -z_4^* & z_3^* \end{pmatrix}^t, \quad \mathsf{w} := \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix}.$$

extended to the whole $\mathcal{O}(S^7)$ *as a* $*$*-algebra morphism.*

$B = \mathcal{O}(S^7)^{co\mathcal{O}(SU(2))}$ *generated by*

$$\alpha := 2(z_1 z_3^* + z_2^* z_4)\,, \quad \beta := 2(z_2 z_3^* - z_1^* z_4)\,, \quad x := z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^* \tag{14}$$

and $*$*-conjugated* α^*, β^*, *with* $x^* = x$.

$\sum z_\mu^* z_\mu = 1$ *implies the four-sphere relation* $\alpha^*\alpha + \beta^*\beta + x^2 = 1$.

For future use the generators satisfy the relations

$$(1-x)z_1 = \alpha z_3 - \beta^* z_4 \qquad (1-x)z_2 = \alpha^* z_4 + \beta z_3$$
$$(x+1)z_3 = \alpha^* z_1 + \beta^* z_2 \qquad (x+1)z_4 = \alpha z_2 - \beta z_1$$

$$S^7 = Spin(5)/SU(2)$$

$$S^4 = Spin(5)/Spin(4) \simeq Spin(5)/SU(2) \times SU(2)$$

The Hopf fibration $S^7 \to S^4$ is a $Spin(5)$-equivariant $SU(2)$-principal bundle. The right-invariant vector fields

$$X \in so(5) \simeq spin(5)$$

on $Spin(5)$ project to the right cosets S^7 and S^4 and generate the $\mathcal{O}(S^7)$-module of vector fields on S^7 and the $\mathcal{O}(S^4)$-module of those on S^4.

Explicit basis:

$$H_1 = \frac{1}{2}(z_1\partial_1 - z_1^*\partial_1^* - z_2\partial_2 + z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*)$$
$$H_2 = \frac{1}{2}(-z_1\partial_1 + z_1^*\partial_1^* + z_2\partial_2 - z_2^*\partial_2^* - z_3\partial_3 + z_3^*\partial_3^* + z_4\partial_4 - z_4^*\partial_4^*) \tag{15}$$

$$E_{10} = \frac{1}{\sqrt{2}}(z_1\partial_3 - z_3^*\partial_1^* - z_4\partial_2 + z_2^*\partial_4^*) \quad E_{-10} = \frac{1}{\sqrt{2}}(z_3\partial_1 - z_1^*\partial_3^* - z_2\partial_4 + z_4^*$$
$$E_{01} = \frac{1}{\sqrt{2}}(z_2\partial_3 - z_3^*\partial_2^* + z_4\partial_1 - z_1^*\partial_4^*) \quad E_{0-1} = \frac{1}{\sqrt{2}}(z_1\partial_4 - z_4^*\partial_1^* + z_3\partial_2 - z_2^*$$
$$E_{11} = -z_4\partial_3 + z_3^*\partial_4^* \quad E_{-1-1} = z_4^*\partial_3^* - z_3\partial_4$$
$$E_{1-1} = -z_1\partial_2 + z_2^*\partial_1^* \quad E_{-11} = -z_2\partial_1 + z_1^*\partial_2^*. \tag{16}$$

$so(5)$ Lie algebra:

$$[H_1, H_2] = 0 \; ; \quad [H_j, E_{\mathsf{r}}] = r_j E_{\mathsf{r}} \; ;$$
$$[E_{\mathsf{r}}, E_{-\mathsf{r}}] = r_1 H_1 + r_2 H_2 \; ; \quad [E_{\mathsf{r}}, E_{\mathsf{s}}] = N_{rs} E_{r+s} \; . \tag{17}$$

The elements H_1, H_2 generators of the Cartan subalgebra, and E_{r} is labelled by

$$\mathsf{r} = (r_1, r_2) \in \Gamma = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\} \; ,$$

one of the eight roots. $N_{rs} = 0$ if r+s is not a root and $N_{rs} \in \{1, -1\}$ otherwise.

$*$-structure: $H_j^* = H_j$ and $E_{\mathsf{r}}^* = E_{-\mathsf{r}}$.

Projected vector fields:

$$\begin{aligned}
H_1^\pi &= \alpha\partial_\alpha - \alpha^*\partial_{\alpha^*} & H_2^\pi &= \beta\partial_\beta - \beta^*\partial_{\beta^*} \\
E_{10}^\pi &= \tfrac{1}{\sqrt{2}}(2x\partial_{\alpha^*} - \alpha\partial_x) & E_{-10}^\pi &= \tfrac{1}{\sqrt{2}}(-2x\partial_\alpha + \alpha^*\partial_x) \\
E_{11}^\pi &= \beta\partial_{\alpha^*} - \alpha\partial_{\beta^*} & E_{-1-1}^\pi &= -\beta^*\partial_\alpha + \alpha^*\partial_\beta \\
E_{01}^\pi &= \tfrac{1}{\sqrt{2}}(2x\partial_{\beta^*} - \beta\partial_x) & E_{0-1}^\pi &= \tfrac{1}{\sqrt{2}}(-2x\partial_\beta + \beta^*\partial_x) \\
E_{1-1}^\pi &= \beta^*\partial_{\alpha^*} - \alpha\partial_\beta & E_{-11}^\pi &= -\beta\partial_\alpha + \alpha^*\partial_{\beta^*}
\end{aligned} \tag{18}$$

$$\mathsf{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7)) = \{X \in \mathsf{Der}(\mathcal{O}(S^7)) \,|\, \delta \circ X = (X \otimes \mathbf{I}) \circ \delta\}. \tag{19}$$

The general H-equivariant derivation is then of the form

$$X = b_1 H_1 + b_2 H_2 + \sum_{\mathsf{r}} b_{\mathsf{r}} E_{\mathsf{r}} \tag{20}$$

for elements $b_j, b_{\mathsf{r}} \in \mathcal{O}(S^4)$. On the generators of $\mathcal{O}(S^7)$

$$X : \mathcal{O}(S^7) \to \mathcal{O}(S^7)\ , \quad \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}^t \mapsto \mathsf{M} \cdot \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \end{pmatrix}^t \tag{21}$$

where M *is the* 4×4 *matrix with entries in $\mathcal{O}(S^4)$*

$$\mathsf{M} = \begin{pmatrix} a_1 & b^*_{1-1} & -b^*_{10} & b_{01} \\ -b_{1-1} & -a_1 & -b^*_{01} & -b_{10} \\ b_{10} & b_{01} & -a_2 & -b_{11} \\ -b^*_{01} & b^*_{10} & b^*_{11} & a_2 \end{pmatrix} , \quad a_1 = \tfrac{1}{2}(b_1 - b_2)\ , a_2 = \tfrac{1}{2}(b_1 + b_2)\ . \tag{22}$$

The derivation (21) *restricts to*

$$X^\pi : \mathcal{O}(S^4) \to \mathcal{O}(S^4)\ , \quad \begin{pmatrix} \alpha & \alpha^* & \beta & \beta^* & x \end{pmatrix}^t \mapsto \mathsf{M}^\pi \begin{pmatrix} \alpha & \alpha^* & \beta & \beta^* & x \end{pmatrix}^t \tag{23}$$

with

$$\mathsf{M}^{\pi} = \begin{pmatrix} b_1 & 0 & b^*_{1-1} & b^*_{11} & \sqrt{2}b^*_{10} \\ 0 & -b_1 & b_{11} & b_{1-1} & \sqrt{2}b_{10} \\ -b_{1-1} & -b^*_{11} & b_2 & 0 & \sqrt{2}b^*_{01} \\ -b_{11} & -b^*_{1-1} & 0 & -b_2 & \sqrt{2}b_{01} \\ -b_{10} & -b^*_{10} & -b_{01} & -b^*_{01} & 0 \end{pmatrix}. \tag{24}$$

The Lie algebra of gauge transformations

We next look for infinitesimal gauge transformations, that is H-equivariant derivations X as in (21) *which are vertical: $X^\pi(b) = 0$, for $b \in \mathcal{O}(S^4)$. These are the kernel of the matrix* M^π *in* (24).

$$
\begin{aligned}
K_1 &:= -iU_2 = 2xH_2 + \beta^*\sqrt{2}E_{01} + \beta\sqrt{2}E_{0-1} \\
K_2 &:= -iU_1 = 2xH_1 + \alpha^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{-10} \\
W_{01} &:= -\tfrac{\sqrt{2}}{2}(W_1 + iW_2) = \sqrt{2}\Big(\beta H_1 + \alpha^* E_{11} + \alpha E_{-11}\Big) \\
W_{0-1} &:= \tfrac{\sqrt{2}}{2}(W_1 - iW_2) = \sqrt{2}\Big(\beta^* H_1 + \alpha^* E_{1-1} + \alpha E_{-1-1}\Big) \\
W_{10} &:= -\tfrac{\sqrt{2}}{2}(W_3 + iW_4) = \sqrt{2}\Big(\alpha H_2 - \beta^* E_{11} + \beta E_{1-1}\Big) \\
W_{-10} &:= \tfrac{\sqrt{2}}{2}(W_3 - iW_4) = \sqrt{2}\Big(\alpha^* H_2 + \beta^* E_{-11} - \beta E_{-1-1}\Big) \\
W_{11} &:= \tfrac{1}{2}(T_1 - iT_2) = 2xE_{11} + \alpha\sqrt{2}E_{01} - \beta\sqrt{2}E_{10} \\
W_{-1-1} &:= -\tfrac{1}{2}(T_1 + iT_2) = 2xE_{-1-1} + \alpha^*\sqrt{2}E_{0-1} - \beta^*\sqrt{2}E_{-10} \\
W_{1-1} &:= -\tfrac{1}{2}(T_3 - iT_4) = -2xE_{1-1} + \beta^*\sqrt{2}E_{10} + \alpha\sqrt{2}E_{0-1} \\
W_{-11} &:= \tfrac{1}{2}(T_3 + iT_4) = -2xE_{-11} + \beta\sqrt{2}E_{-10} + \alpha^*\sqrt{2}E_{01}.
\end{aligned}
\tag{25}
$$

These generators satisfy $K_j(f^*) = -(K_j(f))^*$ and $W_{\mathsf{r}}(f^*) = -(W_{-\mathsf{r}}(f))^*$ for $f \in \mathcal{O}(S^7)$.

Proof: Each vertical derivation, $X = b_1 H_1 + b_2 H_2 + \sum_{\mathsf{r}} b_{\mathsf{r}} E_{\mathsf{r}}$, *with* $b_j, b_{\mathsf{r}} \in \mathcal{O}(S^4)$ *can be expressed as combination of the vertical derivations* K_j, W_{r} *in* (25) *as*

$$X = c_1 K_1 + c_2 K_2 + \sum_{\mathsf{r}} c_{\mathsf{r}} W_{\mathsf{r}}$$

with coefficients $c_1, c_2, c_{\mathsf{r}} \in \mathcal{O}(S^4)$ *given by*

$$\begin{aligned}
c_1 &= \tfrac{1}{4}\Big(2x b_2 + \sqrt{2}\beta\, b_{01} + \sqrt{2}\beta^* b_{0-1}\Big) & c_2 &= \tfrac{1}{4}\Big(2x b_1 + \sqrt{2}\alpha b_{10} + \sqrt{2}\\
c_{01} &= \tfrac{\sqrt{2}}{4}\Big(\beta^* b_1 + \alpha b_{11} + \alpha^* b_{-11}\Big) & c_{0-1} &= \tfrac{\sqrt{2}}{4}\Big(\beta b_1 + \alpha b_{1-1} + \alpha^*\\
c_{10} &= \tfrac{\sqrt{2}}{4}\Big(\alpha^* b_2 - \beta b_{11} + \beta^* b_{1-1}\Big) & c_{-10} &= \tfrac{\sqrt{2}}{4}\Big(\alpha b_2 + \beta b_{-11} - \beta^*\\
c_{11} &= \tfrac{1}{4}\Big(2x b_{11} + \sqrt{2}\alpha^* b_{01} - \sqrt{2}\beta^* b_{10}\Big) & c_{-1-1} &= \tfrac{1}{4}\Big(2x b_{-1-1} + \sqrt{2}\alpha b_0\\
c_{1-1} &= \tfrac{1}{4}\Big(-2x b_{1-1} + \sqrt{2}\beta b_{10} + \sqrt{2}\alpha^* b_{0-1}\Big) & c_{-11} &= \tfrac{1}{4}\Big(-2x b_{-11} + \sqrt{2}\beta^* b
\end{aligned} \tag{26}$$

Remark . The generators in (25) are not independent over the algebra $\mathcal{O}(S^4)$.

$$\begin{aligned}
\beta W_{0-1} - \beta^* W_{01} + \alpha W_{-10} - \alpha^* W_{10} &= 0 \\
-\beta K_2 + \sqrt{2} x W_{01} - \alpha^* W_{11} + \alpha W_{-11} &= 0 \\
-\beta^* K_2 + \sqrt{2} x W_{0-1} - \alpha W_{-1-1} + \alpha^* W_{1-1} &= 0 \\
-\alpha K_1 + \sqrt{2} x W_{10} + \beta^* W_{11} + \beta W_{1-1} &= 0 \\
-\alpha^* K_1 + \sqrt{2} x W_{-10} + \beta W_{-1-1} + \beta^* W_{-11} &= 0 \,.
\end{aligned} \tag{27}$$

Proposition. . *The generators in* (25) *transform under the adjoint representation of* $so(5)$ *with highest weight vector* W_{11}*:*

$$\begin{aligned}
&H_j \triangleright K_l = [H_j, K_l] = 0\,, \quad H_j \triangleright W_{\mathsf{r}} = [H_j, W_{\mathsf{r}}] = r_j W_{\mathsf{r}}\,, \\
&E_{\mathsf{r}} \triangleright K_j = [E_{\mathsf{r}}, K_j] = -r_j W_{\mathsf{r}}\,, \\
&E_{\mathsf{r}} \triangleright W_{\text{-}r} = [E_{\mathsf{r}}, W_{\text{-}r}] = r_1 K_1 + r_2 K_2\,, \quad E_{\mathsf{r}} \triangleright W_{\mathsf{s}} = [E_{\mathsf{r}}, W_{\mathsf{s}}] = N_{rs} W_{r+s}\,,
\end{aligned} \tag{28}$$

A representation theoretical decomposition of $\mathtt{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$

$so(5)$ *irrep. denoted:* $[d(s,n)]$.

highest weight vector of weight $\frac{s}{2}(1,1)+n(1,0)$

dimension: $d(s,n)=\frac{1}{6}(1+s)(1+n)(2+s+n)(3+s+2n)$.

$\mathcal{O}(S^4)$ *decomposes in the sum of spherical harmonics on* S^4

$$\mathcal{O}(S^4)=\bigoplus_{n\in\mathbb{N}_0}[d(0,n)] \tag{29}$$

with $[d(0,n)]$ *the representation of highest weight vector* α^n *of weight* $(n,0)$.

$[5]\otimes[10] \simeq [35]\oplus[10]\oplus[5]$ *with highest weight vectors for these three representations:*

$$\begin{aligned} Z_{21} &= \alpha W_{11}\,, \\ Y_{11} &= \sqrt{2}xW_{11}+\alpha W_{01}-\beta W_{10}\,, \\ X_{10} &= \beta^* W_{11}+\beta W_{1-1}-\alpha K_1+\sqrt{2}xW_{10}, \end{aligned} \tag{30}$$

with the label denoting the value of the corresponding weight.

On S^7 *we have:* X_{10} *vanishes,* $Y_{11}=-\sqrt{2}W_{11}$, Z_{21} *is the* $[35]$.

Proposition. . *The Lie algebra* $\mathtt{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ *decomposes as*

$$\mathtt{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7)) = \bigoplus_{n\in\mathbb{N}_0} [d(2,n)] \,.$$

Here $[d(2,n)]$ *is the representation of* $so(5)$ *as derivations on* $\mathcal{O}(S^7)$ *of highest weight vector* $\alpha^n W_{11}$ *of weight* $(n+1,1)$ *and dimension* $d(2,n) = \frac{1}{2}(n+1)(n+4)(2n+5)$*.*

Gauge Lie algebra: $\mathrm{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))$ *of the generators*

$$[K_1, K_2] = \sqrt{2}(\alpha^* W_{10} - \alpha W_{-10})$$

$$[K_1, W_{01}] = -\sqrt{2}\beta K_2 + 2x W_{01} \qquad [K_1, W_{0-1}] = \sqrt{2}\beta^* K_2 - 2x W_{0-1}$$

$$[K_1, W_{10}] = \sqrt{2}(-\beta^* W_{11} + \beta W_{1-1}) \qquad [K_1, W_{-10}] = \sqrt{2}(\beta W_{-1-1} - \beta^* W_{-11})$$

$$[K_1, W_{11}] = 2x W_{11} - \sqrt{2}\beta W_{10} \qquad [K_1, W_{-1-1}] = -2x W_{-1-1} + \sqrt{2}\beta^* W_{-10}$$

$$[K_1, W_{1-1}] = -2x W_{1-1} + \sqrt{2}\beta^* W_{10} \qquad [K_1, W_{-11}] = 2x W_{-11} - \sqrt{2}\beta W_{-10}$$

$$[K_2, W_{01}] = \sqrt{2}(\alpha^* W_{11} + \alpha W_{-11}) \qquad [K_2, W_{0-1}] = -\sqrt{2}(\alpha^* W_{1-1} + \alpha W_{-1-1})$$

$$[K_2, W_{10}] = 2x W_{10} - \sqrt{2}\alpha K_1 \qquad [K_2, W_{-10}] = -2x W_{-10} + \sqrt{2}\alpha^* K_1$$

$$[K_2, W_{11}] = 2x W_{11} + \sqrt{2}\alpha W_{01} \qquad [K_2, W_{-1-1}] = -2x W_{-1-1} - \sqrt{2}\alpha^* W_{0-1}$$

$$[K_2, W_{1-1}] = 2x W_{1-1} - \sqrt{2}\alpha W_{0-1} \qquad [K_2, W_{-11}] = -2x W_{-11} + \sqrt{2}\alpha^* W_{01}$$

$$[W_{01}, W_{0-1}] = -\sqrt{2}(\alpha^* W_{10} + \alpha W_{-10})$$

$$[W_{01}, W_{10}] = \sqrt{2}(\beta W_{10} - \alpha W_{01}) \qquad [W_{01}, W_{-10}] = -\sqrt{2}(\beta W_{-10} + \alpha^* W_{01})$$

$$[W_{01}, W_{11}] = \sqrt{2}\beta W_{11} \qquad [W_{01}, W_{-1-1}] = -\sqrt{2}\beta W_{-1-1} + \sqrt{2}\alpha^*(K_1 +$$

$$[W_{01}, W_{1-1}] = \sqrt{2}\beta W_{1-1} + \sqrt{2}\alpha(-K_1 + K_2) \qquad [W_{01}, W_{-11}] = -\sqrt{2}\beta W_{-11}$$

$$[W_{0-1}, W_{10}] = \sqrt{2}(\beta^* W_{10} + \alpha W_{0-1}) \qquad [W_{0-1}, W_{-10}] = \sqrt{2}(-\beta^* W_{10} + \alpha^* W_{0-1})$$

$$[W_{0-1}, W_{11}] = \sqrt{2}\beta^* W_{11} - \sqrt{2}\alpha(K_1 + K_2) \qquad [W_{0-1}, W_{-1-1}] = -\sqrt{2}\beta^* W_{-1-1}$$

$$[W_{0-1}, W_{1-1}] = \sqrt{2}\beta^* W_{1-1} \qquad [W_{0-1}, W_{-11}] = -\sqrt{2}\beta^* W_{-11} + \sqrt{2}\alpha^*(K_1 - K_2$$

$$[W_{10}, W_{-10}] = \sqrt{2}(\beta^* W_{01} + \beta W_{0-1})$$

$$[W_{10}, W_{11}] = \sqrt{2}\alpha W_{11} \qquad [W_{10}, W_{-1-1}] = -\sqrt{2}\alpha W_{-1-1} - \sqrt{2}\beta^*(K_1 + K_2)$$

$$[W_{10}, W_{1-1}] = -\sqrt{2}\alpha W_{1-1} \qquad [W_{10}, W_{-11}] = \sqrt{2}\alpha W_{-11} + \sqrt{2}\beta(K_1 - K_2)$$

$$[W_{-10}, W_{11}] = \sqrt{2}\alpha^* W_{11} + \sqrt{2}\beta(K_1 + K_2) \qquad [W_{-10}, W_{-1-1}] = -\sqrt{2}\alpha^* W_{-1-1}$$

$$[W_{-10}, W_{1-1}] = -\sqrt{2}\alpha^* W_{1-1} + \sqrt{2}\beta^*(-K_1 + K_2) \qquad [W_{-10}, W_{-11}] = \sqrt{2}\alpha^* W_{-11}$$

$$[W_{11}, W_{-1-1}] = 2x(K_1 + K_2) + \sqrt{2}\alpha W_{-10} + \sqrt{2}\beta W_{0-1}$$

$$[W_{11}, W_{1-1}] = \sqrt{2}\alpha W_{10} \qquad [W_{11}, W_{-11}] = -\sqrt{2}\beta W_{01}$$

$$[W_{-1-1}, W_{1-1}] = \sqrt{2}\beta^* W_{0-1} \qquad [W_{-1-1}, W_{-11}] = -\sqrt{2}\alpha^* W_{-10}$$

$$[W_{1-1}, W_{-11}] = 2x(-K_1 + K_2) + \sqrt{2}\beta^* W_{01} - \sqrt{2}\alpha W_{-10}$$

TWIST TO braided derivations and infinitesimal gauge transformations

The right invariant vector fields H_1 and H_2 of $Spin(5)$ are the vector fields of a maximal torus $\mathbb{T}^2 \subset Spin(5)$. They define the universal enveloping algebra K of the abelian Lie algebra $[H_1, H_2] = 0$. Twist

$$\mathsf{F} := e^{\pi i\theta(H_1 \otimes H_2 - H_2 \otimes H_1)} \ , \quad \theta \in \mathbb{R}^n \ , \tag{31}$$

with universal R-matrix $\mathsf{R}_\mathsf{F} = \bar{\mathsf{F}}^2$. Twisted product ($\bullet_\mathsf{F} = \bullet_\theta$) :

$$z_1 \bullet_\theta z_3 = e^{\pi i\theta} z_3 \bullet_\theta z_1 \ , \ \ z_1 \bullet_\theta z_4 = e^{-\pi i\theta} z_4 \bullet_\theta z_1$$

$$z_2 \bullet_\theta z_3 = e^{-\pi i\theta} z_3 \bullet_\theta z_2 \ , \ \ z_2 \bullet_\theta z_4 = e^{\pi i\theta} z_4 \bullet_\theta z_2 \ .$$

$\mathcal{O}(SU(2))$-coinvariant subalgebra generated by

$$\alpha := 2(z_1 \bullet_\theta z_3^* + z_2^* \bullet_\theta z_4), \quad \beta := 2(z_2 \bullet_\theta z_3^* - z_1^* \bullet_\theta z_4),$$
$$x := z_1 \bullet_\theta z_1^* + z_2 \bullet_\theta z_2^* - z_3 \bullet_\theta z_3^* - z_4 \bullet_\theta z_4^*. \tag{32}$$

The only nontrivial commutation relations are

$$\alpha \bullet_\theta \beta = e^{-2\pi i\theta} \beta \bullet_\theta \alpha \ , \ \ \alpha \bullet_\theta \beta^* = e^{2\pi i\theta} \beta^* \bullet_\theta \alpha \tag{33}$$

and their complex conjugates.

Braided Lie algebra $so_\theta(5)$ associated with $(K_{\mathsf{F}}, \mathsf{R}_{\mathsf{F}} = \bar{\mathsf{F}}^2)$:

$$[H_1, H_2]_{\mathsf{F}} = [H_1, H_2] = 0 \; ; \quad [H_j, E_{\mathsf{r}}]_{\mathsf{F}} = [H_j, E_{\mathsf{r}}] = r_j E_{\mathsf{r}} \; ;$$
$$[E_{\mathsf{r}}, E_{\mathsf{s}}]_{\mathsf{F}} = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}}[E_{\mathsf{r}}, E_{\mathsf{s}}] = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}} N_{rs} E_{r+s} \; , \tag{34}$$

with $\mathsf{r} \wedge \mathsf{s} := r_1 s_2 - r_2 s_1$.

Module structure in (9)*:*

$$a \cdot_{\mathsf{F}} H_j = a H_j \; , \quad a_{\mathsf{s}} \cdot_{\mathsf{F}} E_{\mathsf{r}} = e^{-\pi i \theta \, \mathsf{s}\wedge\mathsf{r}} \, a_{\mathsf{s}} E_{\mathsf{r}} \; ,$$

for all $a \in \mathcal{O}(S^7_\theta)$ and $a_{\mathsf{s}} \in \mathcal{O}(S^7_\theta)$ eigen-functions of H_j with eigenvalues s_j (being E_{r} eigenvectors of H_j).

Gauge Lie algebra $\left(\mathrm{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))_{\mathsf{F}}, [\ ,\ ]_{\mathsf{F}}, \cdot_{\mathsf{F}}\right)$ associated with $(K_{\mathsf{F}}, \mathsf{R}_{\mathsf{F}})$. It has braided Lie bracket determined on generators:

$$[K_1, K_2]_{\mathsf{F}} = [K_1, K_2] \; ; \quad [K_j, W_{\mathsf{r}}]_{\mathsf{F}} = [K_j, W_{\mathsf{r}}] \; ;$$
$$[W_{\mathsf{r}}, W_{\mathsf{s}}]_{\mathsf{F}} = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}}[W_{\mathsf{r}}, W_{\mathsf{s}}] \; , \tag{35}$$

On generic elements X, X' in the linear span of the generators in (25) *and $b, b' \in \mathcal{O}(S^4)$, the equation* (12) *gives*

$$[b \cdot_{\mathsf{F}} X, b' \cdot_{\mathsf{F}} X']_{\mathsf{F}} = b \cdot_{\mathsf{F}} (\mathsf{R}_{\mathsf{F}\alpha} \triangleright b') \cdot_{\mathsf{F}} [\mathsf{R}_{\mathsf{F}}{}^{\alpha} \triangleright X, X']_{\mathsf{F}} \; . \tag{36}$$

$$\mathsf{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) = \mathcal{D}((\mathsf{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7))_{\mathsf{F}}).$$

Proposition. . *The braided Lie algebra* $\mathsf{Der}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta))$ *of equivariant derivations of the* $\mathcal{O}(SU(2))$*-Hopf–Galois extension* $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ *is generated, as an* $\mathcal{O}(S^4_\theta)$*-module by elements:*

$$\widetilde{H}_j := \mathcal{D}(H_j)\,, \quad \widetilde{E}_{\mathsf{r}} := \mathcal{D}(E_{\mathsf{r}})\ , \quad j = 1, 2\,, \quad \mathsf{r} \in \Gamma \tag{37}$$

with bracket

$$[\widetilde{H}_1, \widetilde{H}_2]_{\mathsf{R}_\mathsf{F}} = \mathcal{D}([H_1, H_2]) = 0\ ; \qquad [\widetilde{H}_j, \widetilde{E}_{\mathsf{r}}]_{\mathsf{R}_\mathsf{F}} = \mathcal{D}([H_j, E_{\mathsf{r}}]) = r_j \widetilde{E}_{\mathsf{r}}\ ;$$
$$[\widetilde{E}_{\mathsf{r}}, \widetilde{E}_{\mathsf{s}}]_{\mathsf{R}_\mathsf{F}} = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}} \mathcal{D}([E_{\mathsf{r}}, E_{\mathsf{s}}]) = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}} N_{rs} \widetilde{E}_{r+s}$$

$\mathsf{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) = \mathcal{D}(\mathsf{aut}_{\mathcal{O}(S^4)}(\mathcal{O}(S^7))_{\mathsf{F}}).$

Proposition. *. The braided Lie algebra* $\mathsf{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ *of infinitesimal gauge transformations of the* $\mathcal{O}(SU(2))$*-Hopf–Galois extension* $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ *is generated, as an* $\mathcal{O}(S^4_\theta)$*-module, by the elements*

$$\widetilde{K}_j := \mathcal{D}(K_j)\,, \quad \widetilde{W}_{\mathsf{r}} := \mathcal{D}(W_{\mathsf{r}})\,, \quad j = 1,2\,, \quad \mathsf{r} \in \Gamma \tag{38}$$

with bracket

$$[\widetilde{K}_1, \widetilde{K}_2]_{\mathsf{R_F}} = \mathcal{D}([K_1, K_2])\;; \quad [\widetilde{K}_j, \widetilde{W}_{\mathsf{r}}]_{\mathsf{R_F}} = \mathcal{D}([K_j, W_{\mathsf{r}}])\;;$$
$$[\widetilde{W}_{\mathsf{r}}, \widetilde{W}_{\mathsf{s}}]_{\mathsf{R_F}} = e^{-i\pi\theta \mathsf{r}\wedge\mathsf{s}}\,\mathcal{D}([W_{\mathsf{r}}, W_{\mathsf{s}}])\;.$$

The braided Lie bracket of generic elements $\widetilde{X}, \widetilde{X}'$ *in* $\mathsf{aut}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta))$ *and* $b, b' \in \mathcal{O}(S^4_\theta)$ *is then given by*

$$[b\bullet_\theta \widetilde{X}, b'\bullet_\theta \widetilde{X}']_{\mathsf{R_F}} = b\bullet_\theta(\mathsf{R_F}_\alpha \triangleright b')\bullet_\theta[\mathsf{R_F}^\alpha \triangleright \widetilde{X}, \widetilde{X}']_{\mathsf{R_F}}\;. \tag{39}$$

The action of any element $\widetilde{W}_{\mathsf{r}}$ *is*

$$\widetilde{W}_{\mathsf{r}}(a_{\mathsf{s}}) = e^{-i\pi\theta\mathsf{r}\wedge\mathsf{s}} W_{\mathsf{r}}(a_{\mathsf{s}})\,, \tag{40}$$

with a braided derivation property

$$\widetilde{W}_{\mathsf{r}}(a_{\mathsf{s}}\bullet_\theta a_{m}) = \widetilde{W}_{\mathsf{r}}(a_{\mathsf{s}})\bullet_\theta a_{m} + e^{-2i\pi\theta\mathsf{r}\wedge\mathsf{s}} a_{\mathsf{s}}\bullet_\theta \widetilde{W}_{\mathsf{r}}(a_{m}). \tag{41}$$

$$[\widetilde{K}_1, \widetilde{K}_2]_{\mathsf{R_F}} = \sqrt{2}(\alpha^* \bullet_\theta \widetilde{W}_{10} - \alpha \bullet_\theta \widetilde{W}_{-10})$$
$$[\widetilde{K}_1, \widetilde{W}_{01}]_{\mathsf{R_F}} = -\sqrt{2}\beta \bullet_\theta \widetilde{K}_2 + 2x \bullet_\theta \widetilde{W}_{01}$$
$$[\widetilde{K}_1, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = -2x \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{10}$$
$$[\widetilde{K}_1, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}e^{-\pi i\theta}\beta \bullet_\theta \widetilde{W}_{1-1} - \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{11}$$
$$[\widetilde{K}_1, \widetilde{W}_{11}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{11} - \sqrt{2}e^{-\pi i\theta}\beta \bullet_\theta \widetilde{W}_{10}$$
$$[\widetilde{K}_2, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{-\pi i\theta}\alpha^* \bullet_\theta \widetilde{W}_{11} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{-11}$$
$$[\widetilde{K}_2, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{1-1} - \sqrt{2}e^{-\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{0-1}$$
$$[\widetilde{K}_2, \widetilde{W}_{10}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{10} - \sqrt{2}\alpha \bullet_\theta \widetilde{K}_1$$
$$[\widetilde{K}_2, \widetilde{W}_{11}]_{\mathsf{R_F}} = 2x \bullet_\theta \widetilde{W}_{11} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{01}$$

$$[\widetilde{W}_{01}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta (\widetilde{K}_2 - \widetilde{K}_1)$$
$$[\widetilde{W}_{01}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{10} - \sqrt{2}e^{\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{01}$$
$$[\widetilde{W}_{01}, \widetilde{W}_{11}]_{\mathsf{R_F}} = \sqrt{2}\beta \bullet_\theta \widetilde{W}_{11}$$
$$[\widetilde{W}_{1-1}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{1-1}$$
$$[\widetilde{W}_{1-1}, \widetilde{W}_{11}]_{\mathsf{R_F}} = -\sqrt{2}e^{-\pi i\theta}\alpha \bullet_\theta \widetilde{W}_{10}$$
$$[\widetilde{W}_{10}, \widetilde{W}_{11}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{11}$$

$$[\widetilde{W}_{-1-1}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-1-1} - \sqrt{2}e^{\pi i\theta}\alpha^* \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2)$$

$$[\widetilde{W}_{-1-1}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = \sqrt{2}e^{-2\pi i\theta}\beta^* \bullet_\theta \widetilde{W}_{0-1}$$

$$[\widetilde{W}_{-1-1}, \widetilde{W}_{10}]_{\mathsf{R_F}} = \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-1-1} + \sqrt{2}e^{\pi i\theta}\beta^* \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2)$$

$$[\widetilde{W}_{-1-1}, \widetilde{W}_{11}]_{\mathsf{R_F}} = -2x \bullet_\theta (\widetilde{K}_1 + \widetilde{K}_2) - \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-10} - \sqrt{2}\beta \bullet_\theta \widetilde{W}_{0-1}$$

$$[\widetilde{W}_{-10}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-10} - \sqrt{2}\alpha^* \bullet_\theta \widetilde{W}_{01}$$

$$[\widetilde{W}_{-10}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = -\sqrt{2}\alpha^* \bullet_\theta \widetilde{W}_{1-1} + \sqrt{2}e^{-\pi i\theta}\beta^* \bullet_\theta (\widetilde{K}_2 - \widetilde{K}_1)$$

$$[\widetilde{W}_{10}, \widetilde{W}_{-10}]_{\mathsf{R_F}} = \sqrt{2}(\beta^* \bullet_\theta \widetilde{W}_{01} + \beta \bullet_\theta \widetilde{W}_{0-1})$$

$$[\widetilde{W}_{-11}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}e^{2\pi i\theta}\beta \bullet_\theta \widetilde{W}_{-11}$$

$$[\widetilde{W}_{0-1}, \widetilde{W}_{01}]_{\mathsf{R_F}} = \sqrt{2}(\alpha^* \bullet_\theta \widetilde{W}_{10} - \alpha \bullet_\theta \widetilde{W}_{-10})$$

$$[\widetilde{W}_{-11}, \widetilde{W}_{1-1}]_{\mathsf{R_F}} = 2x \bullet_\theta (\widetilde{K}_1 - \widetilde{K}_2) - \sqrt{2}\beta^* \bullet_\theta \widetilde{W}_{01} + \sqrt{2}\alpha \bullet_\theta \widetilde{W}_{-10}$$

Atiyah sequences and their splittings (connections)

Let $B = A^{coH} \subseteq A$ *be a* K*-equivariant Hopf–Galois extension* $B = A^{coH} \subseteq A$*, with triangular Hopf algebra* (K, R)*.*

$$\mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) = \{u \in \mathsf{Der}(A) \mid \delta \circ u = (u \otimes \mathbf{I}) \circ \delta\}$$

and then its Lie subalgebra of vertical derivations

$$\mathsf{aut}^{\mathsf{R}}_{B}(A) := \{u \in \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \mid u(b) = 0,\ b \in B\}.$$

Derivations in $\mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A)$ *are* H*-equivariant hence restricts to a derivation on* $B = A^{coH}$*. Associated to* $B = A^{coH} \subseteq A$*, there is the sequence of braided Lie algebras* $\mathsf{aut}^{\mathsf{R}}_{B}(A) \to \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathsf{Der}^{\mathsf{R}}(B)$*.*

When exact,

$$0 \to \mathsf{aut}^{\mathsf{R}}_{B}(A) \to \mathsf{Der}^{\mathsf{R}}_{\mathcal{M}^H}(A) \to \mathsf{Der}^{\mathsf{R}}(B) \to 0 \tag{42}$$

is a version of the Atiyah sequence of a (commutative) principal fibre bundle.

A connection on the bundle can be given as an H*-equivariant splitting of the sequence.*

For the intanton bundle on the sphere S^4_θ we have then the short sequence of braided Lie algebras

$$0 \to \mathsf{aut}^{\mathsf{R_F}}_{\mathcal{O}(S^4_\theta)}(\mathcal{O}(S^7_\theta)) \xrightarrow{\imath} \mathsf{Der}^{\mathsf{R_F}}_{\mathcal{M}^H}(\mathcal{O}(S^7_\theta)) \xrightarrow{\pi} \mathsf{Der}^{\mathsf{R_F}}(\mathcal{O}(S^4_\theta)) \to 0\,. \quad (43)$$

This sequence is exact.

The connection 1-form

The connection on the $SU(2)$-bundle $\mathcal{O}(S^4_\theta) \subset \mathcal{O}(S^7_\theta)$ given by splitting the sequence (43) corresponds to the $su(2)$-valued 1-form on the bundle ω : $\mathsf{Der}^{\mathsf{R_F}}(\mathcal{O}(S^7_\theta)) \to su(2)$ given by

$$\omega = -d\mathsf{u}^* \bullet_\theta \mathsf{u} = \omega_{22} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \omega_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \omega^*_{21} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with 1-forms

$$\omega_{22} = dz_1 \bullet_\theta z^*_1 + dz_2 \bullet_\theta z^*_2 + dz_3 \bullet_\theta z^*_3 + dz_4 \bullet_\theta z^*_4$$

$$\omega_{21} = -dz_1 \bullet_\theta z_2 + dz_2 \bullet_\theta z_1 - dz_3 \bullet_\theta z_4 + dz_4 \bullet_\theta z_3.$$

Thank you